

# AN IMPROVEMENT OF REID INEQUALITY

MOHAMMED HICHEM MORTAD

ABSTRACT. In this short note, we improve the famous Reid Inequality related to linear operators.

## 1. MAIN RESULT

First, assume that readers are familiar with notions and result on  $B(H)$ . We do recall a few definitions and results though:

- (1) Let  $A \in B(H)$ . We say that  $A$  is positive (we then write  $A \geq 0$ ) if
$$\langle Ax, x \rangle \geq 0, \quad \forall x \in H.$$
- (2) For every positive operator  $A \in B(H)$ , there is a unique positive  $B \in B(H)$  such that  $B^2 = A$ . We call  $B$  the positive square root of  $A$ .
- (3) The absolute value of  $A \in B(H)$  is defined to be the (unique) positive square root of the positive operator  $A^*A$ . We denote it by  $|A|$ .
- (4) We recall that  $A \in B(H)$  is called hyponormal if  $AA^* \leq A^*A$ .

The inequality of Reid which first appeared in [4] is recalled next:

**Theorem 1.1.** *Let  $A, K \in B(H)$  such that  $A$  is positive and  $AK$  is self-adjoint. Then*

$$|\langle AKx, x \rangle| \leq \|K\| \langle Ax, x \rangle$$

for all  $x \in H$ .

*Remark.* As shown in e.g. [2], Reid Inequality is equivalent to the operator monotony of the positive square root on the set of positive operators.

Many generalizations of Theorem 1.1 are known in the literature from which we only cite [1] and [2].

In an earlier version of this paper (see [3]), the author showed the following:

**Theorem 1.2.** *Let  $A, K \in B(H)$  such that  $A$  is positive and  $AK$  is normal. Then*

$$|\langle AKx, x \rangle| \leq \|K\| \langle Ax, x \rangle$$

for all  $x \in H$ .

Can we go to  $AK$  being hyponormal? The answer is no as seen next:

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**Example 1.3.** Let  $S$  be the shift operator on  $\ell^2$ . Setting  $A = SS^*$ , we see that  $A \geq 0$ . Now, take  $K = S$  (and so  $\|K\| = 1$ ). It is clear that  $AK = SS^*S = S$  is hyponormal. If Reid Inequality held, then we would have

$$| \langle Sx, x \rangle | \leq \langle SS^*x, x \rangle = \|S^*x\|^2$$

for each  $x \in \ell^2$ . This inequality clearly fails to hold for all  $x$ . Indeed, taking  $x = (2, 1, 0, 0, \dots)$ , we see that

$$| \langle Sx, x \rangle | = 2 \leq \|S^*x\|^2 = 1$$

which is impossible.

The good news is that Reid Inequality can yet be improved as it holds if  $AK$  is co-hyponormal, that is, if  $(AK)^*$  is hyponormal. This comes after a discussion with a fellow student (Mr S. Dehimi):

**Theorem 1.4.** *Let  $A, K \in B(H)$  such that  $A$  is positive and  $(AK)^*$  is hyponormal. Then*

$$| \langle AKx, x \rangle | \leq \|K\| \langle Ax, x \rangle$$

for all  $x \in H$ .

The proof relies on the following result:

**Lemma 1.5.** ([1]) *Let  $A \in B(H)$  be hyponormal. Then*

$$| \langle Ax, x \rangle | \leq \|A\| \langle x, x \rangle.$$

Now, we give the proof of Theorem 1.4.

*Proof.* The inequality is evident when  $K = 0$ . So, assume that  $K \neq 0$ . It is then clear that  $\frac{K}{\|K\|}$  satisfies

$$KK^* \leq \|K\|^2 I.$$

Hence

$$|(AK)^*|^2 = AKK^*A \leq \|K\|^2 A^2$$

or simply  $|(AK)^*| \leq \|K\|A$  after passing to square roots.

Now, for all  $x \in H$

$$| \langle AKx, x \rangle | = | \langle x, (AK)^*x \rangle | = | \overline{\langle (AK)^*x, x \rangle} | = | \langle (AK)^*x, x \rangle |.$$

Since  $(AK)^*$  is hyponormal, Lemma 1.5 combined with  $|(AK)^*| \leq \|K\|A$  give

$$| \langle AKx, x \rangle | = | \langle (AK)^*x, x \rangle | \leq | \langle |(AK)^*|x, x \rangle | \leq \|K\| \langle Ax, x \rangle$$

and this marks the end of the proof.  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ORAN 1, AHMED BEN BELLA, B.P. 1524, EL MENOUAR, ORAN 31000, ALGERIA.

**Mailing address:**

PR MOHAMMED HICHEM MORTAD  
BP 7085 SEDDIKIA ORAN  
31013  
ALGERIA

*E-mail address:* mhmortad@gmail.com, mortad@univ-oran.dz.